



Quantization of abelian varieties: Distributional sections and the transition from Kähler to real polarizations

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Abstract

We study the dependence of geometric quantization of the standard symplectic torus on the choice of invariant polarization. Real and mixed polarizations are interpreted as degenerate complex structures. Using a weak version of the equations of covariant constancy, and the Weil–Brezin expansion to describe distributional sections, we give a unified analytical description of the quantization spaces for all non-negative polarizations. The Blattner–Kostant–Sternberg (BKS) pairing maps between half-form corrected quantization spaces for different polarizations are shown to be transitive and related to an action of $Sp(2g, \mathbb{R})$. Moreover, these maps are shown to be unitary.

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Contents

1. Introduction	3389
2. Preliminaries on the prequantum line bundle and its sections	3390
2.1. The prequantum line bundle L	3390
2.2. Spaces of sections of L^k	3391
2.3. Invariant complex structures and theta functions	3392
2.4. Invariant polarizations	3393

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2.5.	The action of the real symplectic group	3394
3.	A distributional construction of the quantum bundle	3395
3.1.	The weak equations of covariant constancy	3396
3.2.	The half-form correction	3398
3.3.	The Weil representation of the metaplectic group	3400
3.4.	The extended quantum Hilbert bundle	3401
4.	The BKS pairing	3406
4.1.	The BKS pairing on the extended quantum bundle	3406
4.2.	Further properties of the BKS pairing	3408
5.	Tropical theta divisors	3409
	Acknowledgments	3411
	References	3411

1. Introduction

Abelian varieties provide a very rich example of interplay between ideas of algebraic geometry and of geometric quantization. A general problem of great interest in the geometric quantization of a large class of symplectic manifolds is the dependence of quantization on the complex structure [3,16]. (See also [2].)

As is well known, the holomorphic quantization of a symplectic torus, once it is equipped with the structure of abelian variety, produces the space of theta functions. In this case, quantizations in different complex structures yield spaces of theta functions which are naturally unitarily equivalent. This equivalence between holomorphic quantization spaces has been obtained in [3] by means of the parallel transport induced by a heat equation. See also [7,8] for an intrinsically finite-dimensional approach.

In this work, we consider the geometric quantization of the standard symplectic torus of dimension $2g$, $(\mathbb{T}^{2g}, \omega)$, in holomorphic and real non-negative invariant polarizations and for any level $k \in \mathbb{N}$. Due to the group structure of \mathbb{T}^{2g} , and consequent triviality of the tangent bundle, the space of invariant polarizations is canonically identified with the non-negative Lagrangian Grassmannian of a fixed symplectic vector space, \mathbb{C}^{2g} . Thus, one obtains some similarities with the geometric quantization of a vector space [3,21]. For instance, below we construct a Blattner–Kostant–Sternberg (BKS) pairing, between quantization spaces associated with invariant non-negative polarizations, which is transitive, once the half-form correction is included.

The main novel point in our approach is the treatment of the equations of covariant constancy of wave functions, for real and holomorphic polarizations, from a unified analytical stand point, considering the operators of covariant derivation to act on distributional sections of the prequantum line bundle. For real polarizations this yields, as expected, distributional sections supported on Bohr–Sommerfeld fibers; still, it does not always produce the same result as the more traditional cohomological wave function approach to quantization of [26], as can be seen in the case of toric manifolds comparing [4] and [15]. For previous related work on the geometric quantization of abelian varieties see [2], where, in particular, it is explicitly shown that the quantization spaces for any two invariant real polarizations are isomorphic.

In the language of geometric quantization, the space of invariant complex structures corresponds to the space of positive Lagrangian subspaces, which is usually identified with Siegel upper half-space \mathbb{H}_g . To allow the study of the dependence of quantization on the complex structure, including the case of degenerate complex structures on the boundary of \mathbb{H}_g , it is convenient

to consider also another chart that uses the closed Siegel disc, which we denote by \mathbb{D}_g . In this way, we obtain a global parameter also for the non-positive definite part (that becomes identified with $\partial\mathbb{D}_g$) of the Lagrangian Grassmannian.

A crucial contribution for the nice behavior of quantization (or rather of the relation between any two quantizations) on the boundary of the upper half-space is given by the half-form correction. As in many other examples in geometric quantization [29], the half-form is necessary for the unitarity of the pairing maps relating quantizations in different polarizations.

Symmetry groups play an important role in this work. On one hand, the BKS pairing between different invariant polarizations is intimately related with the action of $Sp(2g, \mathbb{R})$ on \mathbb{D}_g . This action does not represent a geometric symmetry of the manifold we are quantizing, but has, instead, an analytical flavor connected to the natural representation of the metaplectic group on $L^2(\mathbb{R}^g)$. Also, one of the most interesting feature of the geometric quantization of \mathbb{T}^{2g} , that is not present in the linear case \mathbb{R}^{2g} (see [21]), besides the appearance of distributional sections of quite different nature over different degenerate Lagrangian subspaces, is the interplay with natural invariance groups of certain data of the prequantization of the torus. Namely, there are natural geometric actions of the integer symplectic group, $Sp(2g, \mathbb{Z})$, and of the group $(\mathbb{Z}/k\mathbb{Z})^{2g}$ on \mathbb{T}^{2g} that leave the holonomies of the prequantum line bundle representing $k\omega$ invariant. While the latter gives rise to the finite Heisenberg group when lifted to the line bundle, the former gives origin, for non-degenerate complex structures, to the classical algebro-geometric theta-transformation formula (see, for example, [6,24,19]). We will address some of these issues in [5].

In [22], Manoliu shows (for even level k) that the BKS pairing maps relating two reducible real polarizations is unitary if one takes into account the half-form correction. In this work, where we consider arbitrary level k , we obtain an analogous result for all holomorphic polarizations, while the real polarizations are included as limiting cases on the boundary of the space of complex structures.

Abelian varieties, therefore, give one more family of symplectic manifolds, in addition to non-compact complex Lie groups (see [14,9]), where half-form corrected holomorphic quantizations in different complex structures, including real polarizations as degenerate cases, can be related by unitary BKS pairing maps [25,29].

The paper is organized as follows. After some preliminaries in Section 2, we use the $Sp(2g, \mathbb{R})$ action in Section 3 to give a unified analytical description of half-form quantization in complex, real, or mixed polarizations, resulting in Theorem 3.10. In Section 4, we show that the BKS pairing maps are unitary and transitive. Section 5 contains a brief illustration of how tropical geometry can be seen to emerge as the complex structure degenerates.

2. Preliminaries on the prequantum line bundle and its sections

2.1. The prequantum line bundle L

Let $(\mathbb{T}^{2g} = \mathbb{R}^{2g}/\mathbb{Z}^{2g}, \omega)$ be the standard even-dimensional torus, with periodic coordinates (x, y) with $x, y \in \mathbb{R}^g$, and the invariant symplectic form given by $\omega = \sum_{i=1}^g dy_i \wedge dx_i$. Consider a prequantization of \mathbb{T}^{2g} given by the line bundle L , representing the cohomology class of the symplectic form, with Hermitian structure and a compatible connection with curvature $-2\pi i\omega$. L is defined by $L = \mathbb{R}^{2g} \times_{\mathbb{Z}^{2g}} \mathbb{C}$, where the \mathbb{Z}^{2g} action is

$$\lambda \cdot (u, \zeta) = (u + \lambda, \alpha(\lambda)e^{-\pi i\omega(u, \lambda)}\zeta), \quad u = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^{2g}, \quad \lambda \in \mathbb{Z}^{2g}, \quad (1)$$

where $\alpha : \mathbb{Z}^{2g} \rightarrow \{\pm 1\} \subseteq U(1)$ is the so-called “canonical” semi-character

$$\alpha(\lambda) = (-1)^{\sum_{j=1}^g \lambda_j \lambda_{g+j}}$$

and ω is identified with symplectic bilinear form

$$\omega(u, v) = \sum_{j=1}^g (u_{g+j} v_j - u_j v_{g+j}) = {}^t u \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} v.$$

The Hermitian structure and connection are defined to be

$$h((u, \zeta), (u, \zeta')) = \zeta \bar{\zeta'} \quad (2)$$

and

$$\nabla s = ds - \pi i s \sum_{j=1}^g (y_j dx_j - x_j dy_j), \quad (3)$$

respectively. Here and below, global sections of L (for example smooth ones, $s \in \Gamma^\infty(L)$) will be identified with sections of the (trivialized) pull-back of L to \mathbb{R}^{2g} , that is, with functions on \mathbb{R}^{2g} that satisfy the appropriate quasi-periodicity conditions,

$$s(u + \lambda) = \alpha(\lambda) e^{-\pi i \omega(u, \lambda)} s(u), \quad \lambda \in \mathbb{Z}^{2g}.$$

Since the trivialization of the bundle is unitary and the connection 1-form purely imaginary, the connection is Hermitian.

2.2. Spaces of sections of L^k

If s is a smooth section of L^k , then

$$e^{k\pi i^t xy} s(x, y)$$

is periodic in x and hence admits a Fourier expansion. This observation leads to an isomorphism given by the Weil–Brezin expansion (also known as Zak expansion) [10] given by

$$\begin{aligned} \Gamma^\infty(L^k) &\rightarrow \prod_{l \in (\mathbb{Z}/k\mathbb{Z})^g} \mathcal{S}(\mathbb{R}^g) \\ s &\mapsto (s)_l(y) := \int_{[0,1]^g} s\left(x, y + \frac{l}{k}\right) e^{k\pi i^t x(y + \frac{l}{k})} e^{-2\pi i^t l x} d^g x \end{aligned} \quad (4)$$

with inverse

$$\prod_{l \in (\mathbb{Z}/k\mathbb{Z})^g} \mathcal{S}(\mathbb{R}^g) \rightarrow \Gamma^\infty(L^k)$$

$$\{(s)_l\}_{l \in (\mathbb{Z}/k\mathbb{Z})^g} \mapsto s(x, y) := e^{-k\pi i^t xy} \sum_{l \in (\mathbb{Z}/k\mathbb{Z})^g} \sum_{m \in \mathbb{Z}^g} (s)_l \left(y - m - \frac{l}{k} \right) e^{2\pi i^t (km+l)x}, \quad (5)$$

where $0 \leq l_j < k$. We will use the round bracket notation $(s)_l$ for the Weil–Brezin coefficients of a section s throughout the paper.

The Weil–Brezin map is an isomorphism between topological vector spaces (of smooth sections and Schwartz functions, respectively), hence it extends to an isomorphism between the dual spaces, that is between the space of distributional sections of the bundle L^k and a product of k^g copies of the space of tempered distributions $\mathcal{S}'(\mathbb{R}^g)$. This map in turn restricts to a unitary isomorphism $\prod_{l \in (\mathbb{Z}/k\mathbb{Z})^g} L^2(\mathbb{R}^g) \leftrightarrow \Gamma_{L^2}(L^k)$, with

$$\langle s, s' \rangle = \sum_{l \in (\mathbb{Z}/k\mathbb{Z})^g} \langle (s)_l, (s')_l \rangle = \sum_{l \in (\mathbb{Z}/k\mathbb{Z})^g} \int_{\mathbb{R}^g} (s)_l \overline{(s')_l}. \quad (6)$$

2.3. Invariant complex structures and theta functions

Invariant complex structures on \mathbb{T}^{2g} are determined by their restriction to the tangent space at any point and hence can be parametrized by matrices in Siegel upper half-space \mathbb{H}_g (see, for instance, [19]). The complex structure on \mathbb{T}^{2g} can be described by its lift to the universal cover \mathbb{R}^{2g} : for any $\Omega \in \mathbb{H}_g$, let Λ_Ω be the lattice $\mathbb{Z}^g \oplus \Omega \mathbb{Z}^g \subset \mathbb{C}^g$, so that the torus is equipped with the structure of an Abelian variety via the smooth isomorphism $\phi_\Omega : \mathbb{T}^{2g} \rightarrow X_\Omega := \mathbb{C}^g / \Lambda_\Omega$ induced by $(x, y \in \mathbb{R}^g)$

$$\mathbb{R}^{2g} \ni u = (x, y) \mapsto z_\Omega := x - \Omega y \in \mathbb{C}^g. \quad (7)$$

(The choice of the sign here, as well as for the symplectic form ω was made so that the action of the symplectic group turns out to be the expected one in all the coordinates). In the coordinates x, y , the complex structure takes the form

$$J_\Omega = \begin{bmatrix} -\Omega_1 \Omega_2^{-1} & \Omega_1 \Omega_2^{-1} \Omega_1 + \Omega_2 \\ -\Omega_2^{-1} & \Omega_2^{-1} \Omega_1 \end{bmatrix}, \quad \Omega = \Omega_1 + i\Omega_2, \quad (8)$$

but we will need this only in the last section. The corresponding holomorphic structure on (L, ∇) is given by the action of the lattice Λ_Ω on $\mathbb{C}^g \times \mathbb{C}$ obtained by combining (1), (7) and the isomorphism

$$\mathbb{R}^{2g} \times \mathbb{C} \ni (u, \zeta) = ((x, y), \zeta) \mapsto (x - \Omega y, e^{\pi i F_\Omega(z_\Omega, z_\Omega)} \zeta) \in \mathbb{C}^g \times \mathbb{C}.$$

We will denote this holomorphic line bundle by $L_\Omega := \mathbb{C}^g \times_{\Lambda_\Omega} \mathbb{C}$. Here F_Ω is the bilinear form

$$F_\Omega(z, w) = -{}^t z (\operatorname{Im} \Omega)^{-1} \operatorname{Im} w$$

or, in terms of real coordinates, $z = x - \Omega y$, $w = u - \Omega v$,

$$F_{\Omega}(x - \Omega y, u - \Omega v) = {}^t(x - \Omega y)v = {}^t x v - {}^t y \Omega v.$$

The quasi-periodicity condition for functions on \mathbb{C}^g to define holomorphic sections of L_{Ω} takes the well-known form corresponding to classical theta functions,

$$\vartheta(z + \lambda) = \alpha(\lambda) e^{2\pi i F_{\Omega}(z, \lambda) + \pi i F_{\Omega}(\lambda, \lambda)} \vartheta(z), \quad \lambda \in \Lambda_{\Omega}.$$

It is well known (see, for example, [19,24]) that a basis of $H^0(X_{\Omega}, L_{\Omega}^k)$ is given by

$$\left\{ \vartheta \begin{bmatrix} \frac{l}{k} \\ 0 \end{bmatrix} (kz, k\Omega) \right\}_{l \in \mathbb{Z}^g / k\mathbb{Z}^g}, \quad (9)$$

which corresponds to the sections

$$\begin{aligned} \vartheta_{\Omega}^l(x, y) &:= e^{-k\pi i F_{\Omega}(x - \Omega y, x - \Omega y)} \vartheta \begin{bmatrix} \frac{l}{k} \\ 0 \end{bmatrix} (kx - k\Omega y, k\Omega) \\ &= e^{-k\pi i {}^t x y} \sum_{m \in \mathbb{Z}^g} e^{k\pi i {}^t (y - m - \frac{l}{k}) \Omega (y - m - \frac{l}{k}) + k2\pi i {}^t (m + \frac{l}{k}) x} \end{aligned} \quad (10)$$

of L^k . Note that the Weil–Brezin coefficients of these sections are the Gaussians

$$(\vartheta_{\Omega}^l)_{l'}(y) = \delta_{ll'} e^{k\pi i {}^t y \Omega y}.$$

2.4. Invariant polarizations

Geometric quantization (for example, of the symplectic torus) can be performed not only using the Kähler structures from the last paragraph, but using the more general notion of a polarization. Here, we describe all invariant polarizations on \mathbb{T}^{2g} .

Let $L\mathbb{T}^{2g} \rightarrow \mathbb{T}^{2g}$ be the bundle of non-negative Lagrangian subspaces of the complexified tangent bundle on the symplectic manifold $(\mathbb{T}^{2g}, \omega)$. A (non-negative) polarization, in the sense of geometric quantization (see, for instance, [29]), is a section of $L\mathbb{T}^{2g}$ that provides an involutive distribution on \mathbb{T}^{2g} ,

$$\text{Pol}(\mathbb{T}^{2g}) \subset \Gamma^{\infty}(L\mathbb{T}^{2g}).$$

In the present case, $L\mathbb{T}^{2g}$ is canonically trivial

$$L\mathbb{T}^{2g} \cong \mathbb{T}^{2g} \times L\mathbb{C}^{2g}$$

(where $L\mathbb{C}^{2g}$ denotes the Grassmannian of non-negative Lagrangian subspaces in \mathbb{C}^{2g} equipped with the standard symplectic form), since $T\mathbb{T}^{2g}$ is (canonically) trivial. A polarization \mathcal{P} , therefore, is simply a function $\mathcal{P} : \mathbb{T}^{2g} \rightarrow L\mathbb{C}^{2g}$, and it is invariant if this function is constant,

$$\text{Pol}^{\mathbb{T}^{2g}}(\mathbb{T}^{2g}) \cong L\mathbb{C}^{2g}.$$

The complex structures from the previous section, parametrized by matrices $\Omega \in \mathbb{H}_g$ in Siegel upper half-space via the coordinates $z_\Omega = x - \Omega y$, correspond to the positive (or Kähler) polarizations $L^+ \mathbb{C}^{2g}$ in $\text{Pol}^{\mathbb{T}^{2g}}(\mathbb{T}^{2g})$,

$$\mathcal{P}_\Omega = \text{span} \left\{ \frac{\partial}{\partial z_\Omega^j} \right\}_{j=1, \dots, g} = \text{span} \left\{ \sum_l \left(\bar{\Omega}_{lk} \frac{\partial}{\partial x_l} + \frac{\partial}{\partial y_k} \right) \right\}_{k=1, \dots, g}.$$

Remark 2.1. Notice that the positive Lagrangian Grassmannian bundle $L^+ \mathbb{T}^{2g} \cong \mathbb{T}^{2g} \times \mathbb{H}_g$ carries also a natural complex structure: equipping every fiber $\mathbb{T}^{2g} \times \{\Omega\}$ with the complex structure from the isomorphism $\mathbb{T}^{2g} \cong \mathbb{C}^g / \Lambda_\Omega$ gives the universal bundle of Abelian varieties (with a marked basis of the first homology) over Siegel upper half-space.

This coordinate chart $\mathbb{H}_g \rightarrow \text{Pol}^{\mathbb{T}^{2g}}(\mathbb{T}^{2g})$ is not convenient for the description of all genuine non-negative polarizations (namely, only those transverse to the polarization spanned by the $\frac{\partial}{\partial x_j}$ directions appear in the closure of \mathbb{H}_g as a space of matrices). The convenient substitute (see, for instance, [20]) for it is the closed Siegel disc \mathbb{D}_g , which is the closure of the image of \mathbb{H}_g under the Cayley transform

$$\mathbb{H}_g \ni \Omega \mapsto \tau = (i - \Omega)(i + \Omega)^{-1} \in \check{\mathbb{D}}_g, \quad (11)$$

with inverse $\Omega = i(1 - \tau)(1 + \tau)^{-1}$. The Siegel disc is a global chart of $L\mathbb{C}^{2g}$, and the parametrization of invariant polarizations now reads

$$\mathcal{P}_\tau = \text{span} \left\{ \sum_l \left(-i(1 - \bar{\tau})_{lk} \frac{\partial}{\partial x_l} + (1 + \bar{\tau})_{lk} \frac{\partial}{\partial y_l} \right) \right\}_{k=1, \dots, g}. \quad (12)$$

Throughout the paper, τ will always denote a point in the closed Siegel disc, while Ω denotes a point in the upper half-space.

Remark 2.2. A polarization is real if and only if τ is unitary. Following [29], we will call a real polarization *reducible*, if its space of leaves is a Hausdorff manifold (or equivalently, in our case, if the leaves are compact). Note that this happens if and only if τ is unitary and its entries lie in $\mathbb{Q}[i]$. Reducible polarizations and the quantizations defined by them (whose elements are supported on leaves along which ∇ has trivial holonomy, the so-called Bohr–Sommerfeld leaves) have been studied in the present context using geometric methods (see [22] and also earlier work by Śniatycki [26]), and will also be investigated in more detail below. Both reducible and non-reducible polarizations are also considered in [2].

2.5. The action of the real symplectic group

Below, we will consider an action of the real symplectic group $Sp(2g, \mathbb{R})$ on the Lagrangian Grassmannian (see [29]), which is intimately related to the construction of the half-form (or metaplectic) correction and the Blattner–Kostant–Sternberg pairing. On the upper half-space chart on the positive Lagrangian Grassmannian, the action is given by the usual fractional linear transformation

$$M(\Omega) = (A\Omega + B)(C\Omega + D)^{-1}, \quad M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in Sp(2g, \mathbb{R}), \quad (13)$$

and it is transitive on \mathbb{H}_g . It extends to all of LC^{2g} (or \mathbb{D}_g), where it becomes transitive on each stratum; in terms of the parameter τ and setting $\tau' := M(\tau)$, the transformation is characterized by the equation

$$((\tau' + 1)(A + iB) + (\tau' - 1)(iC - D))\tau = (\tau' + 1)(A - iB) + (\tau' - 1)(iC + D). \quad (14)$$

In particular, the action of the integer symplectic group $Sp(2g, \mathbb{Z})$ by symplectomorphism of \mathbb{T}^{2g}

$$\mathbb{T}^{2g} \ni \begin{bmatrix} x \\ y \end{bmatrix} \mapsto M \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{T}^{2g},$$

induces an action on polarizations via push-forward given by

$$M_*\mathcal{P}_\Omega = \mathcal{P}_{M(\Omega)}.$$

Remark 2.3. If we write this action on the symplectic coordinates (x, y) in terms of the complex ones, we recover the usual transformation

$$z_\Omega \mapsto z_{\Omega'} = {}^t(C\Omega + D)^{-1}z_\Omega.$$

The metaplectic group (discussed in more detail in Section 3.3 below) is defined as the connected two-fold covering group $Mp(2g, \mathbb{R}) \rightarrow Sp(2g, \mathbb{R})$. It cannot be realized as a matrix group. A convenient description for our purpose (see [28,12]) involves an open subset $U \subset Sp(2g, \mathbb{R})$ that generates the group in two steps, i.e. $U^2 = Sp(2g, \mathbb{R})$; U is parametrized by triples (P, L, Q) of $g \times g$ matrices, with L invertible, P and Q symmetric, via

$$(P, L, Q) \mapsto \begin{bmatrix} PL^{-1} & PL^{-1}Q - {}^tL \\ L^{-1} & L^{-1}Q \end{bmatrix}.$$

The usefulness of this subset U for the description of the metaplectic group lies in the fact that its pre-image \tilde{U} in $Mp(2g, \mathbb{Z})$ consists of two disjoint diffeomorphic copies of U . This gives a workable description of the metaplectic group, as we will see below.

3. A distributional construction of the quantum bundle

In order to treat quantizations using complex, mixed and real polarizations in a uniform way, it is convenient to widen our perspective on the prequantum Hilbert space $\Gamma_{L^2}(L^k)$ of square-integrable sections, and consider instead the whole Gelfand triplet

$$\Gamma^\infty(L^k) \subseteq \Gamma_{L^2}(L^k) \subseteq \Gamma^{-\infty}(L^k).$$

Once we take into account the factors arising from the half-form correction, the pairing on the prequantum Hilbert space Γ_{L^2} and its extension to the Gelfand triplet will, by construction, provide the BKS pairing, as we will see in the next section. The fact that the pairing splits nicely

into a “prequantum factor” and a “half-form factor”, as in the vector space case, is ultimately a consequence of the group structure on $(\mathbb{T}^{2g}, \omega)$. Before studying the pairing, we will define the quantum bundle (or rather, interpret the usual definition, as in [29] or [20]) in the setting just outlined.

Remark 3.1. Similar analytic behavior occurs in the quantization of a flat symplectic vector space, where in particular it also turns out to be necessary to include half-form correction to obtain continuous behavior of the BKS pairing up to the boundary [21]. Also, in the context of toric varieties, a similar convergence to distributional sections occurs when holomorphic polarizations degenerate to the toric real polarization, with fibers the compact Lagrangian tori [4].

Examining the set of differential equations attached to a choice of polarization that singles out the subspaces of covariantly constant sections, we will see that they actually admit a natural weak variant that coincides with the usual equations for square-integrable sections in the case of a complex polarization, but with the advantage that there is a (non-zero) space of solutions also for mixed and real polarizations. It is our aim, in this section, to show that this definition is the natural one from the point of view of the transition from complex to mixed or real polarizations. This is achieved in Theorem 3.10 below.

3.1. The weak equations of covariant constancy

Given an invariant polarization \mathcal{P} , a smooth, at first, local section s is covariantly constant along \mathcal{P} if

$$\forall \xi \in \mathcal{P}: \quad \nabla_{\bar{\xi}} s = ds \cdot \bar{\xi} - k\pi i s(x, y) \sum (y_j dx_j - x_j dy_j) \cdot \bar{\xi} = 0. \quad (15)$$

Note that if the polarization is given by a complex structure, these are just the Cauchy–Riemann equations. In any case, every $\xi \in \mathcal{P}$ defines a continuous linear operator $\nabla_{\bar{\xi}}: \Gamma^\infty(L^k) \rightarrow \Gamma^\infty(L^k)$, and the quantum space associated with a complex polarization \mathcal{P} is given by the intersection of the kernels of these operators.

If the polarization has real directions (and the manifold we are quantizing is compact), there are no non-zero global smooth solutions to Eqs. (15), since any such solution would have to be supported on Bohr–Sommerfeld leaves: smooth sections can be restricted to leaves, and non-trivial holonomy along any non-contractible loop in a leaf forbids the existence of non-zero horizontal sections.

It is, therefore, natural to consider the weak version of the operators (15) acting on distributional sections of L^k . Distributional sections in the quantization of abelian varieties appeared also in [2]. As already mentioned in the introduction, in the case of toric manifolds this approach has the advantage of including contributions from all, even singular, Bohr–Sommerfeld fibers [4]; thus, the dimension of the quantization space in the real toric polarization equals the dimension obtained from Kähler polarization.

Remark 3.2. Locally, the neighborhood of a Bohr–Sommerfeld fiber of \mathbb{T}^{2g} is equivariantly symplectomorphic to a neighborhood of a non-singular Bohr–Sommerfeld fiber of a toric manifold. As seen in [4], each Bohr–Sommerfeld fiber will be the support of a one-dimensional space of polarized distributional sections. Moreover, the explicit form of these sections is described in [4].

For an open subset $U \subset \mathbb{T}^{2g}$, consider first the natural injection

$$\iota : \Gamma^\infty(L^k|_U) \rightarrow \Gamma^{-\infty}(L^k|_U) = (\Gamma_c^\infty(L^k|_U))'$$

$$s \mapsto \iota s(\phi) = \int_U s \phi \frac{\omega^n}{n!}.$$

Then, the definition should make the following diagram commute,

$$\begin{array}{ccc} \Gamma^\infty(L^k) & \xhookrightarrow{\iota} & \Gamma^{-\infty}(L^k) \\ \nabla_\xi \downarrow & & \downarrow \nabla'_\xi \\ \Gamma^\infty(L^k) & \xhookrightarrow{\iota} & \Gamma^{-\infty}(L^k) \end{array}$$

so that the operator ∇'_ξ extends the operator ∇_ξ to distributional sections. Explicitly, on any open set U , for any smooth section $s \in \Gamma_c^\infty(L^k|_U)$ and for any test section $\phi \in \Gamma_c^\infty(L^{-k}|_U)$ with compact support and smooth section $\xi \in \Gamma^\infty(\mathcal{P}|_U)$ of the polarization on U ,

$$\begin{aligned} (\nabla'_\xi \iota s)(\phi) &= \int_U (\nabla_\xi s) \phi \, d^g x \, d^g y \\ &= \int_U \left(ds \cdot \xi - k\pi i s \sum (y_j \, dx_j - x_j \, dy_j) \cdot \xi \right) \phi \, d^g x \, d^g y \\ &= - \int_U s \left(\operatorname{div} \xi \phi + d\phi \cdot \xi + k\pi i \phi \sum (y_j \, dx_j - x_j \, dy_j) \cdot \xi \right) d^g x \, d^g y \\ &= - \int_U s (\operatorname{div} \xi \phi + \nabla_\xi^{-1} \phi) \, d^g x \, d^g y, \end{aligned}$$

where ∇^{-1} stands for the connection on the inverse bundle L^{-k} induced by ∇ .

It is, therefore, natural to define the operation of covariant differentiation of distributional sections, $\forall \sigma \in \Gamma^{-\infty}(L^k|_U)$, $\forall \xi \in \Gamma^\infty(\mathcal{P}|_U)$,

$$(\nabla'_\xi \sigma)(\phi) := -\sigma(\operatorname{div} \xi \phi + \nabla_\xi^{-1} \phi), \quad \forall \phi \in \Gamma_c^\infty(L^{-k}|_U), \quad (16)$$

and to define the quantum space associated with the polarization as the intersections of the kernels of the operators $\nabla'_\xi : \Gamma^{-\infty}(L^k) \rightarrow \Gamma^{-\infty}(L^k)$ for $\xi \in \Gamma^\infty(\mathcal{P})$, as before. For Kähler polarizations, regularity of the Cauchy–Riemann equations (that are the equations of covariant constancy in this case) guarantees that this definition is conservative, that is, one does not find distributional solutions to them which are not holomorphic functions. On regularity of the Cauchy–Riemann equations see, for instance, Chapter 6 in [13] (the argument given there for the case of distributions on \mathbb{C} extends to \mathbb{C}^n), or [18].

3.2. The half-form correction

To define the half-form correction for the torus \mathbb{T}^{2g} we have been considering, let us first recall its definition for the case of a symplectic vector space (V, ω) of real dimension $2g$ (see [29,21]).

As before, we can for simplicity parametrize polarizations (non-negative Lagrangian subspaces $\mathcal{P} \subset V_{\mathbb{C}}$) by the Siegel disc \mathbb{D}_g . Over it, one considers the canonical line bundle $K \subset \mathbb{D}_g \times \Lambda^g V_{\mathbb{C}}^* \rightarrow \mathbb{D}_g$, where each fiber is given by the space of g -forms that vanish upon contraction with the conjugate of any vector in the polarization. We will use the same letter K for the pull-back of this bundle to \mathbb{H}_g . The (indefinite) pairing on the space of g -forms given by

$$\langle \eta, \eta' \rangle \frac{(k\omega)^g}{g!} = 2^{-g} i^g (-1)^{\frac{g(g-1)}{2}} \eta \wedge \overline{\eta'}, \quad (17)$$

induces a pairing between any two fibers of this fibration. Notice that this pairing is invariant under the natural action of the real symplectic group $Sp(V, \omega) \cong Sp(2g, \mathbb{R})$.

Any $\Omega \in \mathbb{H}_g$ defines a positive polarization by specifying a complex coordinate $z_{\Omega} = x - \Omega y$ on V , and $d^g z_{\Omega} := dz_{\Omega}^1 \wedge \cdots \wedge dz_{\Omega}^g$ is a g -form generating the line of K over Ω . A short calculation shows that the pairing comparing the two different fibers over Ω and Ω' is, then,

$$\langle d^g z_{\Omega}, d^g z_{\Omega'} \rangle = \det \frac{1}{2ki} (\Omega - \overline{\Omega'}), \quad (18)$$

and in particular we see that the pairing is positive definite over \mathbb{H}_g .

The half-form (or metaplectic) correction consists in the choice of a square root of K . Since K is trivial, this looks like a trivial operation, but the behavior of the inner product on the square root is an essential analytic ingredient for the BKS pairing. We take advantage of the fact that there is a natural way of defining a specific branch of the square root of a determinant as in (18), given by a Gaussian integral

$$\left(\det \frac{1}{2ki} (\Omega - \overline{\Omega'}) \right)^{-\frac{1}{2}} := \int_{\mathbb{R}^g} e^{-\pi i \frac{t_{\xi}(\Omega - \overline{\Omega'})_{\xi}}{2ki}} d^g \xi = \langle e^{\pi i \frac{t_{\xi} \Omega_{\xi}}{2k}}, e^{\pi i \frac{t_{\xi} \Omega'_{\xi}}{2k}} \rangle_{L^2(\mathbb{R}^g)}.$$

Keeping a traditional, though possibly slightly misleading, notation $\sqrt{d^g z_{\Omega}}$ for the generator of the complex line of the half-form bundle over $\Omega \in \mathbb{H}_g$, we define the pairing via the embedding

$$\alpha \sqrt{d^g z_{\Omega}} \mapsto \frac{1}{\alpha} e^{\pi i \frac{t_{\xi} \Omega_{\xi}}{2k}}, \quad \alpha \in \mathbb{C} \setminus \{0\}, \quad (19)$$

or, more explicitly (and with the sign determined by the Gaussian integral)

$$\langle \sqrt{d^g z_{\Omega}}, \sqrt{d^g z_{\Omega'}} \rangle = \left(\det \frac{1}{2ki} (\Omega - \overline{\Omega'}) \right)^{\frac{1}{2}}. \quad (20)$$

We will see below that this point of view is also very convenient for the description of the action of the metaplectic group on half-forms.

This definition works only over the positive Lagrangian Grassmannian; we need to rescale the trivializing section to describe the half-form bundle over the whole of the non-negative Lagrangian Grassmannian. The canonical line determined by the polarization \mathcal{P}_τ is generated by the g -form

$$d^g(x, y)_\tau = \bigwedge^g [(1 + \tau) dx - i(1 - \tau) dy] = \det(1 + \tau) d^g z_{\Omega(\tau)},$$

where the last equality holds whenever $d^g z_{\Omega(\tau)}$ is defined. The half-form bundle is then described by any of these two trivializations,

$$\mathbb{C} \sqrt{d^g z_{\Omega(\cdot)}} \cong \mathbb{C} \sqrt{d^g(x, y)} \rightarrow \mathbb{D}_g,$$

where the isomorphism is given by

$$\sqrt{d^g z_{\Omega(\tau)}} = (\det(1 + \tau))^{-\frac{1}{2}} \sqrt{d^g(x, y)_\tau}, \quad (21)$$

and the branch of the square root is fixed by demanding it to be 1 for $\tau = 0$.

Let us now address the half-form correction for \mathbb{T}^{2g} . Recall from Section 2.4 that the bundle of non-negative Lagrangian subspaces $L\mathbb{T}^{2g}$ of the complexified tangent bundle is canonically trivialized, $L\mathbb{T}^{2g} \cong \mathbb{T}^{2g} \times L\mathbb{C}^{2g}$. The canonical bundle $K \rightarrow L\mathbb{T}^{2g}$, generated over any point $(x, y, \tau) \in \mathbb{T}^{2g} \times \mathbb{D}_g$ by $d^g(x, y)_\tau$ is again topologically trivial. It is clear that K restricted to $\mathbb{T}^{2g} \times \{\Omega\} \cong X_\Omega$ is the usual canonical bundle $K_\Omega \rightarrow X_\Omega$ of holomorphic g -forms. As before (17), the canonical bundle K comes with a natural Hermitian structure h_K determined by the Liouville form.

As Hermitian bundle, the half-form bundle $\delta \rightarrow L\mathbb{T}^{2g}$ is just the pull-back

$$\begin{array}{ccc} \delta & \longrightarrow & \mathbb{C} \sqrt{d^g(x, y)} \\ \downarrow & & \downarrow \\ L\mathbb{T}^{2g} \cong \mathbb{T}^{2g} \times \mathbb{D}_g & \longrightarrow & \mathbb{D}_g \end{array}$$

A connection on δ is determined as follows. K is equipped with a natural partial connection ∇^{part} given pointwise at $((x, y), \tau)$ by the Lie derivative of complexified g -forms on \mathbb{T}^{2g} at (x, y) along the directions of $\overline{\mathcal{P}}_\tau$. In fact, as K is trivial, ∇^{part} extends to the relative trivial connection ∇^{triv} of the family $L\mathbb{T}^{2g} \rightarrow \mathbb{D}_g$, corresponding to covariant derivatives along the directions of the fibers X_Ω . In particular, the section $(x, y) \mapsto d^g(x, y)_\tau$ is parallel relative to this connection.

The condition imposed on the connection $\tilde{\nabla}$ on the half-form bundle δ is that it satisfies Leibniz's rule

$$\nabla^{\text{triv}}(\mu \otimes \mu') = (\tilde{\nabla}\mu) \otimes \mu' + \mu \otimes (\tilde{\nabla}\mu').$$

Thus, the choice of connection $\tilde{\nabla}$ is determined by the holonomies along $2g$ generators of $H_1(\mathbb{T}^{2g}, \mathbb{R})$, subject to the condition that they square to 1. The Hermitian structure on δ is obtained by taking a consistent square root of the Hermitian structure on K .

Note that $\sqrt{d^g z_\Omega}$ is a parallel section (along the base \mathbb{T}^{2g} , as above) of δ only if $\tilde{\nabla}$ is the trivial connection. Choosing the pair $(\delta, \tilde{\nabla})$ and an invariant holomorphic polarization \mathcal{P}_Ω is equivalent to fixing a specific point χ of order at most two in $\text{Pic}^0(X_\Omega)$, that is, a half-characteristic. If χ is not zero (and does not represent the trivial holomorphic bundle), there are no global holomorphic sections; in particular, $\sqrt{d^g z_\Omega}$, being a smooth trivialization, cannot be parallel along the anti-holomorphic directions. Solving equations of covariant constancy on $L^k \otimes \mathcal{P}_\Omega^* \delta$ gives, then, the space of holomorphic sections $H^0(L^k \otimes \chi)$, i.e. leads to theta functions with half-characteristic.

In the remainder of the paper, for simplicity, we will use the correction $(\delta, \tilde{\nabla})$ corresponding to $\chi = 1 \in \text{Pic}^0(X_\Omega)$. All the results extend to other choices for χ .

3.3. The Weil representation of the metaplectic group

In order to study solutions of the equations of covariant constancy for polarizations on the boundary of \mathbb{D}_g , and in view of the Weil–Brezin expansion, it is convenient to use a natural action of the metaplectic group on $L^2(\mathbb{R}^g)$. Recall from [28] (see also [12,10,11]) that the metaplectic group $Mp(2g, \mathbb{R})$, the connected two-fold cover of $Sp(2g, \mathbb{R})$, can be constructed as a group of unitary operators on the Hilbert space $L^2(\mathbb{R}^g)$, as follows.

Considering real $g \times g$ matrices P, L, Q with P and Q symmetric and L invertible, and an integer $m \bmod 4$ indexing a choice of square root i^m of $\text{sign det } L$, consider the integral operator $S(P, L, Q)_m : \mathcal{S}(\mathbb{R}^g) \rightarrow \mathcal{S}(\mathbb{R}^g)$ given by

$$S(P, L, Q)_m f(u) := i^{-\frac{g}{2}+m} k^{\frac{g}{2}} \sqrt{|\det L|} \int e^{k\pi i({}^t u P u - 2{}^t u {}^t L v + {}^t v Q v)} f(v) d^g v, \quad (22)$$

where we use the notation $i^\phi := e^{\frac{\pi i}{2}\phi}$ for any real number ϕ . These operators are continuous on $\mathcal{S}(\mathbb{R}^g)$, therefore also on $\mathcal{S}'(\mathbb{R}^g)$, and are unitary isomorphisms when restricted to $L^2(\mathbb{R}^g)$.

The metaplectic group is the group of unitary operators generated by all operators of this form. The covering maps each generator $S(P, L, Q)_m$ to the symplectic matrix specified by (P, L, Q) . Note that this projection differs from the one in [12] by an automorphism of $Sp(2g, \mathbb{R})$. Any element in $Mp(2g, \mathbb{R})$ can be represented by the product of two operators of this form. We then have

Lemma 3.3. *Let $\Omega \in \mathbb{H}_g$; then*

$$(S(P, L, Q)_m e^{k\pi i {}^t v \Omega v})(u) = i^m \left(\frac{|\det L|}{\det(\Omega + Q)} \right)^{\frac{1}{2}} e^{k\pi i {}^t u \Omega' u},$$

where $\Omega' \in \mathbb{H}_g$ is given by

$$\Omega' = P - {}^t L(\Omega + Q)^{-1} L = \begin{bmatrix} P L^{-1} & P L^{-1} Q - {}^t L \\ L^{-1} & L^{-1} Q \end{bmatrix}(\Omega).$$

Proof. Applying the definitions,

$$S(P, L, Q)_m e^{k\pi i {}^t v \Omega v}(u) = i^{-\frac{g}{2}+m} k^{\frac{g}{2}} \sqrt{|\det L|} \int e^{k\pi i({}^t u P u - 2{}^t u {}^t L v + {}^t v(\Omega + Q)v)} d^g v$$

$$= i^{-\frac{g}{2}+m} k^{\frac{g}{2}} \sqrt{|\det L|} \sqrt{\frac{\pi^g}{\det -k\pi i(\Omega + Q)}} \\ \times e^{k\pi i^t u P u} e^{-k\pi i^t u^t L(\Omega + Q)^{-1} L u},$$

from which the assertion follows. \square

Remark 3.4. Note that the sign of the square root in the statement $\sqrt{\det(\Omega + Q)}$ is determined by evaluation of the Gaussian integral in the proof. We will not need to specify it explicitly.

The action of $Sp(2g, \mathbb{R})$ on the Lagrangian Grassmannian, given by $\Omega \mapsto M(\Omega)$, lifts to K , where one obtains the ordinary pull-back of g -forms

$$(M^*)^{-1} d^g z_\Omega = \det(C\Omega + D) d^g z_{\Omega'}, \quad (23)$$

for $M \in Sp(2g, \mathbb{R})$ as in (13). By the construction of the half-form bundle δ and the identification of the Hermitian metric (19) on it, the unitary representation of the metaplectic group that lifts (23) is then given by

$$S(P, L, Q)_m \sqrt{d^g z_\Omega} = i^{-m} |\det L|^{-\frac{1}{2}} \sqrt{\det(\Omega + Q)} \sqrt{d^g z_{\Omega'}}, \quad (24)$$

where $S(P, L, Q)_m$ is any generator of $Mp(2g, \mathbb{R})$ and $\Omega' = M(\Omega)$, with $M = (P, L, Q)$.

From this follows immediately

Proposition 3.5. *The product of the action of the metaplectic group $Mp(2g, \mathbb{R})$ on $\Gamma^{-\infty}(L^k)$ with the natural action on δ descends to an action of $Sp(2g, \mathbb{R})$ on $\Gamma^{-\infty}(L^k) \otimes \sqrt{d^g(x, y)_\tau} \rightarrow \mathbb{D}_g$ that lifts the symplectic action on the Siegel disc.*

3.4. The extended quantum Hilbert bundle

As explained above, since we consider real polarizations as limits of holomorphic polarizations, and since for the real polarizations there are no smooth solutions of the equations of covariant constancy, we use the weak version of these equations and look for solutions in the space of distributional sections $\Gamma^{-\infty}(L^k)$. This is particularly simple to do in terms of the Weil–Brezin expansion in (4), and making use of the parameter $\tau \in \mathbb{D}_g$ for the polarization in Section 2.4.

Let $\tilde{\mathcal{Q}}_\tau$ be the space of distributional sections $\sigma \in \Gamma^{-\infty}(L^k)$ that are solutions of the weak equations of covariant constancy (16) with respect to the polarization \mathcal{P}_τ . These spaces naturally form a bundle of vector subspaces in the bundle of distributional sections,

$$\tilde{\mathcal{Q}} \subset \Gamma^{-\infty}(L^k) \times \mathbb{D}_g \rightarrow \mathbb{D}_g.$$

Lemma 3.6. *Under the identification of global sections of $\Gamma^{-\infty}(L^k)$ with elements in $S'(\mathbb{R}^g)^g$ via the Weil–Brezin transform, the equations of covariant constancy defining $\tilde{\mathcal{Q}}_\tau$ become independent for each of the g components $S'(\mathbb{R}^g)$, and are identical on all of them.*

Explicitly, using the first-order differential operators $\mathcal{E}_\tau : \mathcal{S}'(\mathbb{R}^g) \rightarrow \mathcal{S}'(\mathbb{R}^g)^g$ defined by

$$\mathcal{E}_\tau = (\tau + 1) \frac{\partial}{\partial y} - 2k\pi(\tau - 1)y, \quad \tau \in \mathbb{D}_g, \quad (25)$$

we have

$$\sigma \in \tilde{\mathcal{Q}}_\tau \Leftrightarrow \forall l = 1, \dots, g: (\sigma)_l \in \text{Ker } \mathcal{E}_\tau.$$

Proof. Since all the operators involved are continuous, it suffices to check the identity on the dense subspace of smooth sections $s \in \Gamma^\infty(L^k)$. Using (3) and (5) we obtain

$$(\nabla_\xi s)_l(y) = {}^t\beta \frac{\partial}{\partial y}(s)_l(y) + k2\pi i {}^t\alpha y(s)_l(y),$$

where $\xi = {}^t\alpha \frac{\partial}{\partial x} + {}^t\beta \frac{\partial}{\partial y}$ is any constant vector field. From (12) we immediately get the lemma. Note that we get the same equation for each Weil–Brezin coefficient separately. \square

Definition 3.7. The quantum Hilbert bundle \mathcal{Q} over the space of invariant polarizations of $(\mathbb{T}^{2g}, \omega)$ is defined as

$$\mathcal{Q} := \tilde{\mathcal{Q}} \otimes \mathbb{C} \sqrt{d^g(x, y)} \rightarrow \mathbb{D}_g,$$

where $\mathbb{C} \sqrt{d^g(x, y)} \rightarrow \mathbb{D}_g$ is the half-form bundle. To simplify the notation we will write $\mathcal{Q}_\tau = \tilde{\mathcal{Q}}_\tau \sqrt{d^g(x, y)_\tau}$.

Recall that due to regularity of the Cauchy–Riemann equations the elements of the quantum Hilbert space \mathcal{Q}_Ω over a point Ω in the interior $\check{\mathbb{D}}_g \cong \mathbb{H}_g$ of the Lagrangian Grassmannian are given by the holomorphic sections of L^k , with the holomorphic structure determined by Ω , tensored with the half-form correction $\mathbb{C} \sqrt{d^g z_\Omega}$,

$$\mathcal{Q}_\Omega \cong H^0(X_\Omega, L^k_\Omega) \sqrt{d^g z_\Omega}.$$

Definition 3.7 extends the bundle of quantizations over the Siegel upper half-space to the boundary of the Siegel disc \mathbb{D}_g , proceeding as depicted in the following diagram:

$$\begin{array}{ccc} \Gamma^\infty(L^k) \sqrt{d^g z_\cdot} & \hookrightarrow & \Gamma^{-\infty}(L^k) \sqrt{d^g(x, y)_{\tau(\cdot)}} \\ \cup & & \cup \\ \mathcal{Q} & \hookrightarrow & \mathcal{Q} \\ \downarrow & & \downarrow \\ \mathbb{H}_g \cong \check{\mathbb{D}}_g & \hookrightarrow & \mathbb{D}_g \end{array}$$

The quantum bundle is naturally viewed as a sub-bundle of the trivial (infinite-dimensional) bundle of smooth (on the left part of the diagram) or distributional (on the right part of the

diagram) sections of the half-form corrected prequantum bundle. By abuse of notation, we will usually not distinguish the spaces of smooth sections from the distributions they naturally define by integrating against Liouville measure.

First, we prove that the dimension of the spaces \mathcal{Q}_τ (or $\tilde{\mathcal{Q}}_\tau$) is independent of $\tau \in \mathbb{D}_g$. (Recovering a result also in [2].)

Lemma 3.8. *For all $\tau \in \mathbb{D}_g$ with $\det(1 + \tau) \neq 0$,*

$$\dim \mathcal{Q}_\tau = k^g.$$

Explicitly, the corresponding subspace $\tilde{\mathcal{Q}}_\tau$ of $\Gamma^{-\infty}(L^k)$ is spanned by the sections with Weil–Brezin coefficients

$$(\tilde{\vartheta}_\tau^l)_{l'} := \delta_{l,l'} \det(1 + \tau)^{-\frac{1}{2}} e^{k\pi i l' y \Omega(\tau)y} \in \mathcal{S}'(\mathbb{R}^g), \quad (26)$$

where the branch of the square root is the same one as in (21).

Proof. Note that the Weil–Brezin coefficients in the statement of the lemma are well-defined elements in $\mathcal{S}'(\mathbb{R}^g)$ even when the imaginary part of Ω is only positive semi-definite, and that the dependence on τ is continuous in this case.

By Lemma 3.6 we have to identify the kernel of the operators \mathcal{E}_τ . Since $(1 + \tau)$ is invertible and

$$\mathcal{E}_\tau = (\tau + 1) \frac{\partial}{\partial y} - 2k\pi(\tau - 1)y = (\tau + 1) \left(\frac{\partial}{\partial y} - 2k\pi i \Omega y \right),$$

the kernel of \mathcal{E}_τ on distributions is one-dimensional and is spanned by the Gaussian in the statement of the lemma (see e.g. the proof of Theorem 7.6.1 in [17]). \square

In order to treat the points where $\det(1 + \tau) = 0$, we now show that the $Sp(2g, \mathbb{R})$ action preserves the kernels of the operators \mathcal{E}_τ .

Consider the operators $\mathcal{E}_\tau \otimes 1$ acting on $\mathcal{S}'(\mathbb{R}^g) \otimes \delta$, where δ is the half-form bundle. Recall that, from Proposition 3.5, the action of the metaplectic group $Mp(2g, \mathbb{R})$ on $\Gamma^{-\infty}(L^k) \otimes \delta$ actually descends to an action of $Sp(2g, \mathbb{R})$. Note also that the group $Sp(2g, \mathbb{R})$ acts diagonally on the g factors of $\mathcal{S}'(\mathbb{R}^g)$ in the Weil–Brezin expansion.

In fact, we now show that the action of $Sp(2g, \mathbb{R})$ lifts to the Hilbert quantum bundle \mathcal{Q} .

Proposition 3.9. *Let, as above, $M = (P, L, Q) \in Sp(2g, \mathbb{R})$ be a matrix with $g \times g$ block entries A, B, C, D , and $\tau' = M(\tau)$; furthermore, consider the matrix*

$$X_{\tau'} := \frac{1}{2}((\tau' + 1)(A + iB) + (\tau' - 1)(iC - D))$$

acting on a vector of g elements of $\mathcal{S}'(\mathbb{R}^g)$. Then

$$(\mathcal{E}_{\tau'} \otimes 1) \circ M = X_{\tau'} \circ M \circ (\mathcal{E}_\tau \otimes 1).$$

In particular, the symplectic transformation M preserves the kernels of the family of operators \mathcal{E}_t , which define the bundle \mathcal{Q} .

Proof. To shorten the expressions, set

$$e(u, v) := e^{k\pi i(t^t u P u - 2^t u^t L v + ^t v Q v)};$$

and calculate, using an arbitrary lifting $M_m \in Mp(2g, \mathbb{Z})$ and Proposition 3.5,

$$\begin{aligned} & (\mathcal{E}_{\tau'} \otimes 1 \circ M)(f(u) \sqrt{d^g(x, y)}_\tau) \\ &= \left((\tau' + 1) \frac{\partial M_m f}{\partial u} - 2k\pi(\tau' - 1)u M_m f(u) \right) M_m(\sqrt{d^g(x, y)}_\tau) \\ &= \int \left((\tau' + 1) \frac{\partial e(u, v)}{\partial u} - 2k\pi(\tau' - 1)u e(u, v) \right) f(v) d^g v \sqrt{d^g(x, y)}_{\tau'} \\ &= \int ((\tau' + 1)2k\pi i(Pu - ^t L v) - 2k\pi(\tau' - 1)u) e(u, v) f(v) d^g v \sqrt{d^g(x, y)}_{\tau'}. \end{aligned}$$

Using the fact that

$$u e(u, v) = L^{-1} Q v e(u, v) - \frac{1}{2k\pi i} L^{-1} \frac{\partial e(u, v)}{\partial v}$$

and integrating by parts, we find

$$\begin{aligned} & (\mathcal{E}_{\tau'} \otimes 1 \circ M)(f(u) \sqrt{d^g(x, y)}_\tau) \\ &= 2k\pi \int \left[i(\tau' + 1) \left(P \left(L^{-1} Q v e(u, v) - \frac{1}{2k\pi i} L^{-1} \frac{\partial e(u, v)}{\partial v} \right) - ^t L v e(u, v) \right) \right. \\ &\quad \left. - (\tau' - 1) \left(L^{-1} Q v e(u, v) - \frac{1}{2k\pi i} L^{-1} \frac{\partial e(u, v)}{\partial v} \right) \right] f(v) d^g v \sqrt{d^g(x, y)}_{\tau'} \\ &= 2k\pi \int \left[i(\tau' + 1) \left(P \left(L^{-1} Q v f(v) + \frac{1}{2k\pi i} L^{-1} \frac{\partial f}{\partial v} \right) - ^t L v f(v) \right) \right. \\ &\quad \left. - (\tau' - 1) \left(L^{-1} Q v f(v) + \frac{1}{2k\pi i} L^{-1} \frac{\partial f}{\partial v} \right) \right] e(u, v) d^g v \sqrt{d^g(x, y)}_{\tau'} \\ &= \int \left[((\tau' + 1)A + i(\tau' - 1)C) \frac{\partial f}{\partial v} + 2k\pi(i(\tau' + 1)B - (\tau' - 1)D) v f(v) \right] \\ &\quad \times e(u, v) d^g v \sqrt{d^g(x, y)}_{\tau'}. \end{aligned}$$

Therefore, to complete the proof it suffices to show that

$$\begin{aligned} & (\tau' + 1)A + i(\tau' - 1)C = X_{\tau'}(1 + \tau), \\ & i(\tau' + 1)B - (\tau' - 1)D = X_{\tau'}(1 - \tau), \end{aligned}$$

but these two equations are equivalent to the following

$$\begin{aligned}(\tau' + 1)(A + iB) + (\tau' - 1)(iC - D) &= 2X_{\tau'}, \\ (\tau' + 1)(A - iB) + (\tau' - 1)(iC + D) &= 2X_{\tau'}\tau,\end{aligned}$$

the first of which is the definition of $X_{\tau'}$, and the second one is precisely (14). \square

Now, we can use the metaplectic group action to show that the rank of the bundle is constant over all of the Siegel disc \mathbb{D}_g . Consider the following section of $\mathcal{Q} \rightarrow \mathbb{H}_g \cong \check{\mathbb{D}}_g$,

$$\sigma_{\Omega}^l := \vartheta_{\Omega}^l \sqrt{d^g z_{\Omega}} = \tilde{\vartheta}_{\tau(\Omega)}^l \sqrt{d^g(x, y)_{\tau(\Omega)}} \in H^0(X_{\Omega}, L_{\Omega}^k) \sqrt{d^g z_{\Omega}},$$

where accordingly

$$\tilde{\vartheta}_{\tau}^l = \det(1 + \tau)^{-\frac{1}{2}} \vartheta_{\Omega(\tau)}^l,$$

with ϑ_{Ω}^l defined by (10) for $\Omega \in \mathbb{H}_g$ and where the branch of the square root is the natural one, as in (21).

Putting things together, we obtain

Theorem 3.10. *Each section $\tilde{\vartheta}_{\tau}^l$ extends continuously to a map $\mathbb{D}_g \rightarrow \Gamma^{-\infty}(L^k)$. In particular, $\tau \mapsto \{\sigma_{\tau}^l\}_{l \in \mathbb{Z}^g/k\mathbb{Z}^g}$ is a set of global sections of \mathcal{Q} , which is therefore trivialized and of rank k^g .*

Furthermore, the elements of the trivializing frame $\{\sigma_{\tau}^l\}_{l \in \mathbb{Z}^g/k\mathbb{Z}^g}$ for \mathcal{Q} are invariant under the $Sp(2g, \mathbb{R})$ action.

Proof. From Proposition 5.4.7 in [29], for any $\tau \in \partial\mathbb{D}_g$, there exists a symplectic transformation $M \in Sp(2g, \mathbb{R})$ such that $M(\tau)$ does not have eigenvalue -1 . Since M^{-1} acts continuously on \mathcal{Q} by the previous proposition, the assertion follows.

The invariance of the sections σ_{τ}^l under the $Sp(2g, \mathbb{R})$ action is an immediate consequence of Lemma 3.3 and of (24). \square

Remark 3.11. For the case of reducible polarizations, the dimension k^g coincides with the one obtained by considering Śniatycki's quantization procedure [26,22] where one considers smooth sections of the restriction of L^k to Bohr–Sommerfeld fibers. In fact, solutions of the weak equations of covariant constancy must be supported along fibers of trivial holonomy, and polarized sections are linear combinations of Dirac delta distributions with phase variation along the Bohr–Sommerfeld fibers.

Example 3.12. For the case of the horizontal polarization, given by $\tau = -1$, the basis $\{\sigma_{-1}^l\}_{l \in (\mathbb{Z}/k\mathbb{Z})^g}$ consists of distributional sections, each supported on a single compact Bohr–Sommerfeld fiber of the vertical polarization: this follows immediately from the equations of covariant constancy written in terms of the Weil–Brezin coefficients. Since $\tau = -1$, the operator \mathcal{E}_{-1} in (25) is just multiplication by $4k\pi y$, and its kernel is generated by the Dirac delta distribution supported at $y = 0$. Therefore, from (5), the corresponding sections σ_{-1}^l (each of which has a single non-zero Weil–Brezin coefficient) will be supported on the points with the components of y being congruent to $\frac{l}{k}$. These points define the k^g Bohr–Sommerfeld fibers of

the vertical polarization. Since each basis element is supported on a single Bohr–Sommerfeld fiber, $\{\sigma_{-1}^l\}_{l \in (\mathbb{Z}/k\mathbb{Z})^g}$ is a so-called Bohr–Sommerfeld basis [27] for \mathcal{Q}_{-1} .

In order to describe Bohr–Sommerfeld basis for other reducible polarizations, it is convenient to study a natural geometric action of the integer symplectic group $Sp(2g, \mathbb{Z})$. For $M \in Sp(2g, \mathbb{Z})$ and τ a reducible polarization, this action transforms Bohr–Sommerfeld leaves of the polarization τ into Bohr–Sommerfeld leaves of the reducible polarization $M(\tau)$. We will describe and study some properties of this action, and of its lift to the quantum bundle, in a forthcoming paper [5]. In fact, that study amounts to a study of the algebro-geometric theta transformation formula from a symplectic point of view.

In the following section, we study the Hilbert space structure on the fibers of the bundle $\mathcal{Q} \rightarrow \mathbb{D}_g$, and how they are related for different fibers.

4. The BKS pairing

4.1. The BKS pairing on the extended quantum bundle

By the very construction of the quantum bundle \mathcal{Q} , over the interior of the Siegel disc the BKS pairing is already implemented as the product of the pairing of square integrable sections with the pairing of half-forms. Explicitly, one has

Theorem 4.1. *For $\Omega, \Omega' \in \mathbb{H}_g$, the BKS pairing is given by*

$$\langle \vartheta_{\Omega}^l \otimes \sqrt{d^g z_{\Omega}}, \vartheta_{\Omega'}^{l'} \otimes \sqrt{d^g z_{\Omega'}} \rangle_{BKS} = 2^{-\frac{g}{2}} k^{-g} \delta_{l, l'}. \quad (27)$$

Proof. We have to calculate the pairing $\langle \vartheta_{\Omega}^l, \vartheta_{\Omega'}^{l'} \rangle$ in the prequantum Hilbert space; from the unitary isomorphism (6) between the space of square integrable sections of L^k and $(L^2(\mathbb{R}^g))^k$ that restricts to the Weil–Brezin expansion on smooth sections, and the Gaussians in (26) it is clear that this gives

$$\langle \vartheta_{\Omega}^l, \vartheta_{\Omega'}^{l'} \rangle = \delta_{l, l'}(ki)^{-\frac{g}{2}} (\det(\overline{\Omega'} - \Omega))^{-\frac{1}{2}}, \quad (28)$$

which, combined with (20) proves this result. \square

Note that from Theorem 3.10, the corresponding BKS pairing map

$$B_{\Omega, \Omega'} : \mathcal{Q}_{\Omega} \rightarrow \mathcal{Q}_{\Omega'},$$

defined by

$$\langle B_{\Omega, \Omega'} \sigma, \sigma' \rangle_{\mathcal{Q}_{\Omega'}} = \langle \sigma, \sigma' \rangle_{BKS}, \quad \forall \sigma \in \mathcal{Q}_{\Omega}, \sigma' \in \mathcal{Q}_{\Omega'},$$

corresponds to the action on Weil–Brezin coefficients of the element of $Sp(2g, \mathbb{R})$ relating Ω and Ω' . It follows that the family of indexed basis $\{\sigma_{\Omega}^l\}_{l \in \mathbb{Z}^g/k\mathbb{Z}^g}$ is parallel with respect to the family of pairing maps.

Moreover, from Theorem 3.10, the BKS pairing can be extended continuously to the boundary of \mathbb{D}_g , so that, $\forall \tau, \tau' \in \mathbb{D}_g$, we can define unitary BKS pairing maps

$$\begin{aligned} B_{\tau, \tau'} : \mathcal{Q}_\tau &\rightarrow \mathcal{Q}_{\tau'} \\ B_{\tau, \tau'}(\sigma_\tau^l) &= \sigma_{\tau'}^l, \quad \forall l \in \mathbb{Z}^g / k\mathbb{Z}^g. \end{aligned} \quad (29)$$

Remark 4.2. The pairing between elements of \mathcal{Q}_τ and $\mathcal{Q}_{\tau'}$, for $\tau \in \mathbb{D}_g$ and $\tau' \in \check{\mathbb{D}}_g$, as in Theorem 4.1 is still given by evaluating the distributional sections in \mathcal{Q}_τ on the conjugate of the smooth sections in $\mathcal{Q}_{\tau'}$ and multiplying with the factor that arises from half-form correction. In this situation, for $\sigma \in \mathcal{Q}_\tau$ and $\sigma' \in \mathcal{Q}_{\tau'}$,

$$\langle \sigma, \sigma' \rangle_{BKS} = \sigma(\overline{\sigma'}) = \overline{\sigma'}\sigma(1),$$

where in the last term the distribution $\overline{\sigma'}\sigma$ is evaluated at the constant function 1.

Therefore, we have

Corollary 4.3. *The family of indexed bases*

$$\tau \mapsto \{\sigma_\tau^l\}_{l \in \mathbb{Z}^g / k\mathbb{Z}^g} \subseteq \mathcal{Q}_\tau, \quad \tau \in \mathbb{D}_g,$$

is parallel with respect to the transitive family of unitary pairing maps

$$B_{\tau, \tau'} : \mathcal{Q}_\tau \rightarrow \mathcal{Q}_{\tau'},$$

$\tau, \tau' \in \mathbb{D}_g$, defined by

$$\langle B_{\tau, \tau'}\sigma, \sigma' \rangle_{\mathcal{Q}_{\tau'}} = \langle \sigma, \sigma' \rangle_{BKS}, \quad \forall \sigma \in \mathcal{Q}_\tau, \sigma' \in \mathcal{Q}_{\tau'}.$$

These results show that symplectic tori also provide examples of symplectic manifolds where quantizations in different polarizations can be related by transitive unitary BKS pairing maps, if the half-form correction is included.

Remark 4.4. From a different but related perspective, the unitary maps $B_{\Omega, \Omega'}$ for $\Omega, \Omega' \in \mathbb{H}_g$ can also be realized as coherent state transformations associated to different Kähler structures on the complex Lie group $(\mathbb{C}^*)^g$. (See [7] and also [14,9].)

Remark 4.5. These results extend straightforwardly to non-principally polarized abelian varieties. An interesting application is then to the moduli space of rank n semistable (degree zero) vector bundles on an elliptic curve E_Ω , for $\Omega \in \mathbb{H}_1$. This moduli space is isomorphic to \mathbb{P}^{n-1} and can be realized as a quotient of the (non-principally polarized) abelian variety $M = E_\Omega \otimes \check{\Lambda}_R$ by the Weyl group of $SL_n(\mathbb{C})$, $\mathbb{P}^{n-1} \cong M/W$, where Λ_R is the corresponding co-root lattice. Non-abelian theta functions in genus one are then realized as W anti-invariant theta functions on M . The unitary equivalence between spaces of non-abelian theta functions, in genus one, associated with different complex structures studied in [3,8], can therefore be formulated equivalently in terms of BKS pairing maps.

4.2. Further properties of the BKS pairing

Recall that the subgroup of translations of X_τ preserving the holomorphic structure on L^k has a central extension, given by the “finite” Heisenberg group H_k which is given by

$$H_k = \{(\lambda, (a, b)) \mid \lambda \in U(1), (a, b) \in (\mathbb{Z}/k\mathbb{Z})^{2g}\},$$

with the group law

$$(\lambda, (a, b))(\lambda', (a', b')) = (\lambda\lambda' e^{\frac{\pi i}{k}(ab' - ba')}, a + a', b + b').$$

H_k acts naturally on $H^0(L^k) \cong \mathcal{Q}_\tau$ by

$$(\lambda, (a, b))\sigma(x, y) = \lambda e^{\pi i \omega((x, y), (a, b))} \sigma\left(x - \frac{a}{k}, y - \frac{b}{k}\right).$$

This well-known natural algebro-geometric irreducible unitary representation of H_k on \mathcal{Q}_τ , which is unique up to isomorphism, for $\tau \in \mathbb{D}_g$ (see for example Section 6.4 of [6], or [24]) is given in the parallel basis by

$$\begin{aligned} (\lambda, (a, 0))\sigma_\tau^l &= \lambda e^{-2\pi i l a/k} \sigma_\tau^l, \\ (\lambda, (0, b))\sigma_\tau^l &= \lambda \sigma_\tau^{l+b}. \end{aligned}$$

It is then natural to consider this representation for all $\tau \in \mathbb{D}_g$, including degenerate points on $\partial\mathbb{D}_g$.

Remark 4.6. From Corollary 4.3 it follows immediately that the unitary BKS pairing maps $B_{\tau\tau'} : \mathcal{Q}_\tau \rightarrow \mathcal{Q}_{\tau'}$ intertwine the canonical representations of H_k .

We will now show that the BKS pairing between transverse real polarizations is, as expected, given by an intersection pairing. Let $\tau \in \partial\mathbb{D}_g$ define a real reducible polarization \mathcal{P}_τ and let $BS \subset \mathbb{T}^{2g}$ be the union of its Bohr–Sommerfeld fibers. Recall from Remark 3.2 that if $\sigma \in \mathcal{Q}_\tau$ then the support of σ is contained in BS .

Proposition 4.7. *Let $\tau, \tau' \in \partial\mathbb{D}_g$ be such that $\mathcal{P}_\tau, \mathcal{P}_{\tau'}$ are real, reducible and transverse. If $\sigma \in \mathcal{Q}_\tau, \sigma' \in \mathcal{Q}_{\tau'}$ then the pairing $\langle \sigma, \sigma' \rangle_{BKS}$ is obtained by evaluating a distribution supported in $BS \cap BS'$ on the constant function 1.*

Proof. From Remark 3.2 and [4], one has that σ is a linear combination of smooth phases multiplying codimension g Dirac delta distributions supported in BS , and similarly for σ' . Since, \mathcal{P}_τ and $\mathcal{P}_{\tau'}$ are transverse, Theorem 8.2.4 and Example 8.2.11 of [17] guarantee that the product $\sigma \overline{\sigma'}$ gives a well-defined distribution on \mathbb{T}^{2g} , supported in $BS \cap BS'$. From the continuity of the product (see Theorem 8.2.4 in [17]) and Remark 4.2, it is clear that $\sigma \overline{\sigma'}(1) = \langle \sigma, \sigma' \rangle_{BKS}$. \square

Remark 4.8. In [22], Manoliu has considered the quantization of the standard symplectic torus in real reducible polarizations at even level k . She introduces geometrically an intersection pairing between quantization spaces for two reducible (transverse) polarizations via the intersection points of the respective Bohr–Sommerfeld leaves. After including the half-form correction these pairings are unitary.

Remark 4.9. Another particularly interesting point on $\partial\mathbb{D}_g$ is the one corresponding to the vertical polarization, $\tau = 1$. In this case, the period matrix $\Omega = 0$ and the theta functions become linear combinations of Dirac delta distributions supported on Bohr–Sommerfeld fibers. Remarkably, the linear combination coefficients are precisely given by the modular transformation S -matrix for characters of the affine Lie algebra $u(1)_k$ [7]. Thus, in terms of Bohr–Sommerfeld basis, the BKS pairing map $B_{-1,1}$ is exactly represented by this modular transformation matrix.

5. Tropical theta divisors

Tropical geometry is known to appear in the description of degeneration data of boundary points in the compactification of the moduli space of polarized abelian varieties [1]. In [23], the authors consider tropical curves, their Jacobians as well as tropical theta functions and tropical theta divisors. Tropical geometry is also known to arise when one degenerates Kähler metrics. In this section, we will comment briefly on the appearance of tropical geometry as one takes the complex structure of \mathbb{T}^{2g} to the boundary of \mathbb{D}_g .

Recall that tropical geometry is the algebraic geometry of curves over the tropical semi-field $\mathbb{R} \cup \{-\infty\}$, where the operations are defined by $x \oplus y = \max\{x, y\}$ and $x \odot y = x + y$, for $x, y \in \mathbb{R} \cup \{-\infty\}$. The evaluation of a polynomial in n variables over this ring defines a piecewise linear map from $(\mathbb{R} \cup \{-\infty\})^n \rightarrow \mathbb{R} \cup \{-\infty\}$. The associated tropical affine variety is then given by the set of points where this function is not smooth, which is a piecewise linear set.

Let us now describe several examples of metric degeneration of Kähler structure at a real reducible polarization. Namely, for computational simplicity, we will choose the point $\tau = -1$. We will see that the outcome depends considerably on the path in \mathbb{D}_g through which τ approaches the point $\tau = -1 \in \partial\mathbb{D}_g$.

From the expression (8) for the complex structure defined by $\Omega \in \mathbb{H}_g$, it is clear that the corresponding Kähler metric $\gamma_\Omega = \omega(\cdot, J_\Omega \cdot)$ is given by the matrix

$$\gamma_\Omega = \begin{bmatrix} \Omega_2^{-1} & -\Omega_2^{-1} \Omega_1 \\ -\Omega_1 \Omega_2^{-1} & \Omega_1 \Omega_2^{-1} \Omega_1 + \Omega_2 \end{bmatrix}. \quad (30)$$

Example 5.1. For the standard hyperbolic metric in \mathbb{H}_g invariant under $Sp(2g, \mathbb{R})$, the geodesics rays going to the point at infinity corresponding to $\tau = -1$ are given by

$$\Omega(s) = B^t A + i A e^{2s\Lambda} A, \quad s > 0,$$

where Λ is a positive diagonal matrix, $A \in GL_n(\mathbb{R})$ and $B^t A$ is symmetric. Then, from (30),

$$\gamma_{\Omega(s)} = \begin{bmatrix} {}^t A^{-1} e^{-2s\Lambda} A^{-1} & -{}^t A^{-1} e^{-2s\Lambda} B \\ -B e^{-2s\Lambda} A^{-1} & B e^{-2s\Lambda} B + A e^{2s\Lambda} A \end{bmatrix}.$$

We see that, after rescaling the metric appropriately, as $s \rightarrow \infty$ at least g dimensions collapse. In fact, the number of surviving dimensions equals the multiplicity of the largest eigenvalue of A .

Let us therefore consider the case when $A = \lambda 1$. Then the rescaled metrics $e^{-2s\lambda} \gamma_{\Omega(s)}$ converge in the Gromov–Hausdorff sense,

$$(\mathbb{T}^{2g}, e^{-2s\lambda} \gamma_{\Omega(s)}) \rightarrow (\mathbb{T}^g, A^t A).$$

One may ask how theta divisors behave under this metric degeneration. For the sake of simplicity, consider for instance the theta divisor $V(\vartheta_\Omega)$ for level $k = 1$. Let $u = e^{2s\lambda}$. The absolute values of the terms of the series defining $\vartheta_{\Omega(s)}$ are

$$a_m(y) = e^{-\pi u^t(y-m)A^t A(y-m)}, \quad m \in \mathbb{Z}^g.$$

Call $m \in \mathbb{Z}^g$ a lattice neighbor of y if

$$\forall m' \in \mathbb{Z}^g \setminus \{m\}: \quad \|y - m\|_{A^t A} \leq \|y - m'\|_{A^t A}.$$

As $A^t A$ is positive definite, it is easy to see that for any point y_0 which does not have at least two distinct lattice neighbors, the theta function cannot equal zero for s large enough and any value of x , i.e.

$$y_0 \notin \mu(V(\vartheta_{\Omega(s)})),$$

where μ stands for the group-valued moment map

$$\mathbb{T}^{2g} \ni (x, y) \mapsto \mu(x, y) = y \in \mathbb{T}^g.$$

The theta divisor then approaches the tropical theta divisor of [1,23] (see Section 5.2 of [23]), defined by the non-smoothness locus of the functions

$$\max_{m \in \mathbb{Z}^g} \{-^t(y-m)A^t A(y-m)\} \quad \text{or} \quad \max_{m \in \mathbb{Z}^g} \{-^t m A^t A m + 2^t y A^t A m\}.$$

Note that this set depends only on the limit metric and on the location of the lattice points which define the Bohr–Sommerfeld fiber. This behavior of theta divisors as one approaches the boundary of \mathbb{D}_g is consistent with the fact that the theta function approaches a distributional section supported on the Bohr–Sommerfeld fiber, so that its zeroes are away from this fiber.

In general, the limiting behavior of the theta divisor will not be related to the limit metric in such a simple manner as above.

Example 5.2. Let $s > 0$. Considering a half-line of complex structures $\Omega + s\dot{\Omega}$ with $\Omega, \dot{\Omega} \in \mathbb{H}_g$ and $s \rightarrow \infty$, the rescaled metrics $\frac{1}{s} \gamma_{\Omega + s\dot{\Omega}}$ converge in the Gromov–Hausdorff sense,

$$\left(\mathbb{T}^{2g}, \frac{1}{s} \gamma_{\Omega + s\dot{\Omega}} \right) \rightarrow (\mathbb{T}^g, \dot{\Omega}_1 \dot{\Omega}_2^{-1} \dot{\Omega}_1 + \dot{\Omega}_2).$$

Note that the point on the boundary of \mathbb{D}_g we are approaching (or the real polarization) is still the same, $\tau = -1$.

A computation analogous to the one in Example 5.1 shows that the limit of the theta divisors is the same as above for the metric $\dot{\Omega}_2$. This coincides with the limit metric only in the case $\dot{\Omega}_1 = 0$, which is when the half-line is a (reparametrized) geodesic.

Example 5.3. For higher level $k > 1$, a particular tropical theta divisor is given in an analogous way by the non-smoothness locus of

$$y \mapsto \max_{m \in \mathbb{Z}^g} \{-^t m \dot{\Omega}_2 m + 2k^t y \dot{\Omega}_2 m\}.$$

This piecewise linear object, which is equidistant from Bohr–Sommerfeld fibers, is obtained by degeneration of the divisor of the theta function $\sum_{l \in \mathbb{Z}^g / k\mathbb{Z}^g} \sigma_\tau^l$, which is invariant by the subgroup $\mathbb{Z}^g / k\mathbb{Z}^g \subset H_k$ generated by the elements of the form $(1, (0, b))$. (See Section 4.2.) Similar objects, the geometric quantization amoebas of [4], appear in the geometric quantization of toric manifolds, where also such a particular tropical divisor, equidistant from Bohr–Sommerfeld points in the polytope, (asymptotically) selects a particular degenerating holomorphic section.

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